Computing Circuits and Sign Vectors with Applications to Linear Inequality Systems and Reaction Networks

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Motivation

- $A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m$
- $x \geq 0$: $x_i \geq 0$ for all i

$\boxed{\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}}$ $\quad x \in \mathbb{R}^n$?

→ Compute solution (e.g. polyhedra)

- No solution? Certificate? → Dual system

Theorem (Farkas 1902)

Either $\boxed{\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}}$ *or* $\boxed{\begin{aligned} A^\top u &\geq 0 \\ b^\top u &< 0 \end{aligned}}$ *has a solution.*

- Theorem of the alternative

$$0 \leq x^\top (A^\top u) = (Ax)^\top u = b^\top u < 0 \ ⚡$$

→ u certifies unsolvability!

- Other certificate? Sparse certificate? → Approach via circuits

Approach & Overview

- Results/theory as general as possible
  - → General (ordered) fields, integral domains
  - → General implementation
  - → Exact computation

SageMath Packages

Circuits of a Subspace — ElementaryVectors

Linear Inequality Systems — CertLin

Sign Vectors — SignVectors

Oriented Matroids — SignVectors

Chemical Reaction Networks — SignCRN

Circuits of a Subspace

- Field $\mathcal{K}$
- For $x \in \mathcal{K}^n$,

$$\operatorname{supp} x = \{i \in [n] : x_i \neq 0\}.$$

$$[n] = \{1, \ldots, n\}$$

- Subspace $\mathcal{V}$ of $\mathcal{K}^n$

$$\operatorname{supp} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \{1, 3\}$$

Definition

$v \in \mathcal{V} \setminus \{0\}$ is circuit (of $\mathcal{V}$) if for all $x \in \mathcal{V} \setminus \{0\}$,

$$\operatorname{supp} x \subseteq \operatorname{supp} v \implies \operatorname{supp} x = \operatorname{supp} v.$$

→ Unique up to multiples
  - Circuits with $\operatorname{supp} x = \operatorname{supp} y$ → x, y multiples
  - Finitely many!

Computing Circuits via Maximal Minors

$$P = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & -3 \end{pmatrix} \to P(\{2,3\}) = \det \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} = 4$$

J	$P(J)$
$\{1,2\}$	1
$\{1,3\}$	2
$\{1,4\}$	-3
$\{2,3\}$	4
$\{2,4\}$	-6
$\{3,4\}$	0

- $P = (p_1, \dots, p_n) \in \mathcal{K}^{r \times n}$ with rank r
- $J = \{j_1, \dots, j_r\} \subseteq [n]$ with $j_1 < \cdots < j_r$

→ Maximal minor: $P(J) = \det(p_{j_1}, \dots, p_{j_r})$

Theorem

For $I \subseteq [n]$ with $|I| = r + 1$, define $v \in \mathcal{K}^n$ by

$$v_i = \begin{cases} \varepsilon(i)\, P(I \setminus \{i\}), & \textit{if } i \in I, \\ 0, & \textit{else.} \end{cases}$$

If $v \neq 0$, then v is circuit of $\ker P$.

- $\varepsilon(i) = \pm 1$

I	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$
$v(I)$	$\begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 3 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 3 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 6 \\ 4 \end{pmatrix}$

- For integral domains!
- How many? → $\binom{n}{r+1}$

Unique Enumeration of Circuits

- Reuse zero minor $\rightarrow$ Multiple
- Mark zero minors
- Lexicographic ordering

$\rightarrow$ Algorithm [AR26a]

- Using Grassmann-Plücker identity

Theorem (Grassmann-Plücker Identity)

$K, I \subseteq [n]$ *with* $|K| = r - 1$ *and* $|I| = r + 1$. *Then,*

$$\sum_{i \in I \setminus K} \varepsilon(i)\, P(K \cup \{i\})\, P(I \setminus \{i\}) = 0.$$

- $\varepsilon(i) = \pm 1$
- For commutative rings

Duality of Maximal Minors

- Integral domain $\mathcal{R}$
- $P \in \mathcal{R}^{r \times n}$ with rank r
- $Q \in \mathcal{R}^{n-r \times n}$ with $\ker Q = \mathrm{row}\, P$

Theorem

There are $\alpha, \beta \in \mathcal{R} \setminus \{0\}$ such that for all $J \subseteq [n]$ with $|J| = r$:

$$\alpha P(J) = \beta \varepsilon(J) Q([n] \setminus J).$$

- $\varepsilon(J) = \pm 1$

→ Compute cocircuits!
  - Circuits of $\mathrm{row}\, P$

Example

$$P = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & -3 \end{pmatrix}$$

$$Q = \begin{pmatrix} 4 & -2 & 1 & 0 \\ -6 & 3 & 0 & 1 \end{pmatrix}$$

J	$P(J)$	$Q([n] \setminus J)$
$\{1,2\}$	1	1
$\{1,3\}$	2	-2
$\{1,4\}$	-3	-3
$\{2,3\}$	4	4
$\{2,4\}$	-6	6
$\{3,4\}$	0	0

Package ElementaryVectors

```
sage: from elementary_vectors import *
sage: P = matrix([[1, 2, 0, 0], [0, 1, 2, -3]])
sage: P
[ 1  2  0  0]
[ 0  1  2 -3]
sage: circuits(P)
[(4, -2, 1, 0), (-6, 3, 0, 1), (0, 0, 3, 2)]
sage: cocircuits(P) # circuits in row space
[(0, -1, -2, 3), (1, 0, -4, 6), (2, 4, 0, 0)]

sage: var('a, b')
(a, b)
sage: P = matrix([[1, 2, a, 0], [0, 1, 2, b]])
sage: P
[1 2 a 0]
[0 1 2 b]
sage: circuits(P)
[(-a + 4, -2, 1, 0), (2*b, -b, 0, 1), (a*b, 0, -b, 2),
 (0, a*b, -2*b, -a + 4)]
```

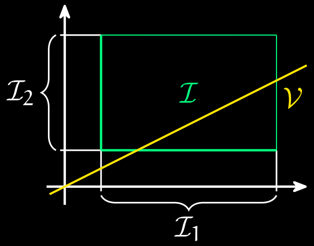
Package ElementaryVectors: Runtime

- Maximal minors
  - → Compute once
    - Hash table
  - → No precomputation
    - Compute single circuits

Table: Runtime (in s) for computing circuits

Ring	Size	Circuits	Time	Note
$\mathbb{Z}$	5×15	$5 \cdot 10^3$	0.1	
	5×20	$4 \cdot 10^4$	0.7	
	7×25	$1 \cdot 10^6$	30	4TI2: 180s
	12×25	$5 \cdot 10^6$	300	4TI2: > 2h
$\overline{\mathbb{Q}}$	4×15	$3 \cdot 10^3$	5	
	5×20	$4 \cdot 10^4$	30	only supports

Minty's Lemma



- Subspace $\mathcal{V} \subseteq \mathbb{R}^m$
- Intervals $\mathcal{I} = \mathcal{I}_1 \times \cdots \times \mathcal{I}_m \subseteq \mathbb{R}^m$
- $y \in \mathcal{V} \cap \mathcal{I}$?

Theorem (Rockafellar 1969, Minty 1974)

(1) *Either* $y \in \mathcal{V} \cap \mathcal{I}$ *exists,*

(2) *or circuit* v *of* $\mathcal{V}^\perp$ *exists with* $z^\top v \neq 0$ *for all* $z \in \mathcal{I}$.

- $\mathbb{R} \rightarrow$ Ordered field $\mathcal{K}$
- Finitely many circuits
  - $\rightarrow$ Certify nonexistence of y
- Linear inequality systems?
  - $\rightarrow$ $\mathcal{V} = \operatorname{im} M$ with $M \in \mathcal{K}^{m \times n}$
    - $y \in \mathcal{V} \cap \mathcal{I} \rightarrow Mx \in \mathcal{I}$

Example

Equivalent are:

- $$\begin{aligned} 1 \leq x_1 - x_2 &< 2 \\ x_2 &> 3 \end{aligned}$$
- $$\underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}_{M} x \in \underbrace{[1, 2) \times (3, \infty)}_{\mathcal{I}}$$

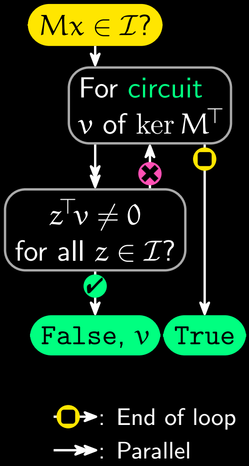
Minty's Lemma: Algorithm

Corollary

(1) *Either* $Mx \in \mathcal{I}$ *has a solution,*

(2) *or circuit* v *of* $\ker M^\top$ *exists with* $z^\top v \neq 0$ *for all* $z \in \mathcal{I}$.

- Certify unsolvability!
- Certify solvability?
  - → Apply approach to dual system!
  - → Compute solution! Via circuits?



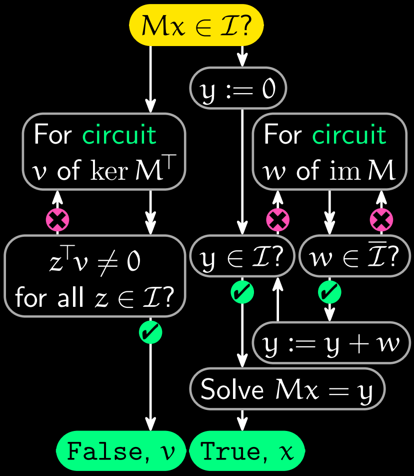
Computing Solutions via Circuits

- "Homogeneous" intervals $\mathcal{I}_i \in \{\{0\}, [0,\infty), (0,\infty)\}$
- Closure $\overline{\mathcal{I}}$ with $\overline{\mathcal{I}_i} \in \{\{0\}, [0,\infty)\}$

Corollary (Based on Rockafellar)

If $Mx \in \mathcal{I}$ has a solution, there are circuits $\mathcal{C}$ of $\operatorname{im} M$ with

$$\sum_{w \in \mathcal{C} \cap \overline{\mathcal{I}}} w = Mx \in \mathcal{I}.$$



→ Run algorithms in parallel

→ Random circuits

→ One hash table for minors

- For ordered integral domain
- General systems?
  - → Homogenize

↠: Parallel

CertLin: Live Demo

[Backup]

CertLin: Runtime

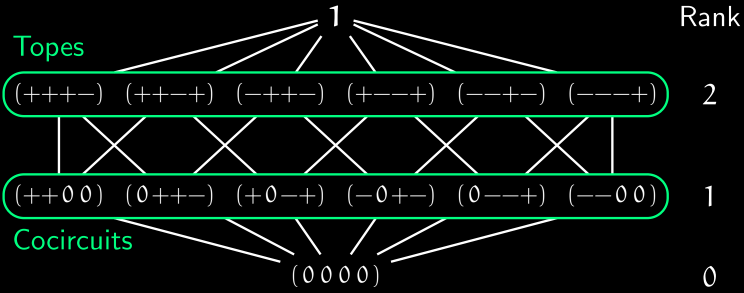
- $Mx \in \mathcal{I} \longrightarrow$ Polyhedron?
  - Homogenize
  - Dual system
  - Compute point in polyhedron $\longrightarrow$ Certify unsolvability

Table: Runtime (in s) for certifying unsolvability

Ring	Size	Minors	CertLin	Polyhedra
$\mathbb{Z}$	20×10	10^5	0.001 – 2	0.7 – 50
	30×15	10^8	0.03 – 3	crash
	60×30	10^{17}	0.5 – 30	–
	100×50	10^{29}	10 – 100	–
$\overline{\mathbb{Q}} \cap \mathbb{R}$	10×5	10^2	0.1 – 0.3	–
	16×8	10^4	0.4 – 4	–

Sign Vectors & Oriented Matroids

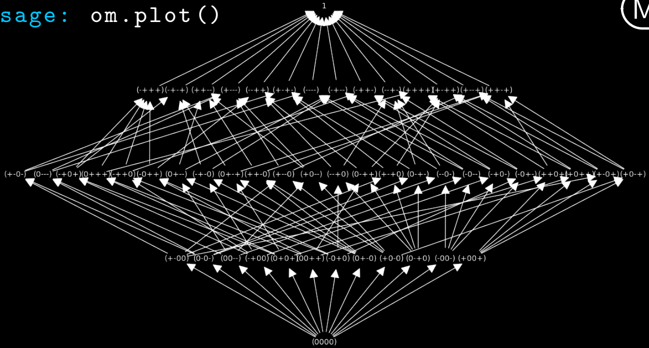
- Sign vector: Element in $\{-,0,+\}^n$ $\qquad \operatorname{sign}\begin{pmatrix}-1\\0\\2\end{pmatrix} = (-\,0\,+)$
- Oriented matroid: Sign vectors of real subspace
  $\rightarrow$ Fulfills covector axioms
- Partial order on $\{-,0,+\}^n$ by $0 \preceq -,+$
- Artificial top element 1 $\rightarrow$ Big face lattice

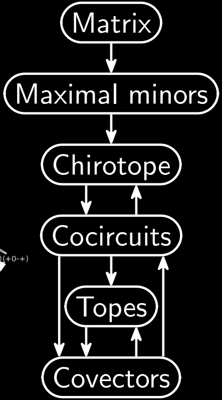


Package SignVectors

- Software presentation [Aic26]
- Fukuda 1991: Topes → All covectors by rank
    - Extend: Construct big face lattice

```
sage: from sign_vectors import *
sage: P = random_matrix(ZZ, 3, 4)
sage: om = OrientedMatroid(P)
sage: om.plot()
```






```
sage: om.chirotope() # signs of maximal minors
[+, +, -, 0]
```

Chemical Reaction Networks

- Weighted directed graph with rate constants $k_{12}, k_{21} > 0$:

$$1 \underset{k_{21}}{\overset{k_{12}}{\rightleftarrows}} 2$$

- Stoichiometric & kinetic complexes: $a, b \in \mathbb{R}$.

$$1A + 1B \rightleftarrows C$$

$$aA + bB \rightleftarrows C$$

- ODE system for concentrations x:

$$\frac{dx}{dt} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{Y} \underbrace{\begin{pmatrix} -k_{12} & k_{21} \\ k_{12} & -k_{21} \end{pmatrix}}_{A_k} \underbrace{\begin{pmatrix} x_A^a x_B^b \\ x_C \end{pmatrix}}_{x^{\widetilde{Y}}} = Y \underbrace{A_k x^{\widetilde{Y}}}_{=0?}$$

- Positive complex-balanced equilibrium (CBE): $x \in \mathbb{R}^3_{>0}$ with $A_k x^{\widetilde{Y}} = 0$
- Study CBEs via subspaces: $\mathcal{S} = \operatorname{im} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \widetilde{\mathcal{S}} = \operatorname{im} \begin{pmatrix} -a \\ -b \\ 1 \end{pmatrix}$

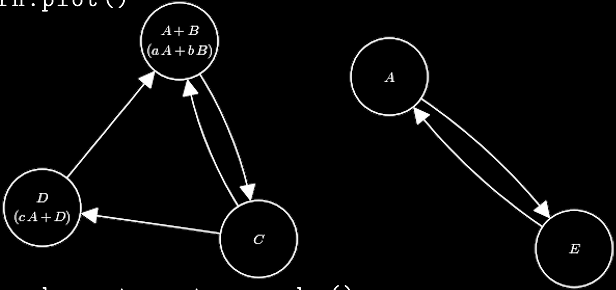
Theorem (Müller, Regensburger 2012)

There is at most one positive CBE [...] iff

- $\operatorname{sign}(\mathcal{S}) \cap \operatorname{sign}(\widetilde{\mathcal{S}}^{\perp}) = \{0\}$. $\rightarrow$ *Use chirotope/maximal minors*

Package SignCRN

```
sage: from sign_crn import *
sage: var("a, b, c")
(a, b, c)
sage: species("A, B, C, D, E")
(A, B, C, D, E)
sage: rn = ReactionNetwork()
sage: rn.add_complexes([(0, A+B, a*A+b*B), (1, C)...])
sage: rn.add_reactions([(0, 1), (1, 0), (1, 2)...])
sage: rn.plot()
```




```
sage: rn.has_at_most_one_cbe()
[{a >= 0, b >= 0, a - c >= 0}]
```

Unique Existence of CBE

Theorem (Müller, Hofbauer, Regensburger 2019)

There is unique positive CBE [...] iff

(1) $\mathrm{sign}(\mathcal{S}) \cap \mathrm{sign}(\widetilde{\mathcal{S}}^{\perp}) = \{0\}$,

(2) *for every nonzero* $\widetilde{\pi} \in \mathrm{sign}(\widetilde{\mathcal{S}}^{\perp})_{\oplus}$,
there is nonzero $\pi \in \mathrm{sign}(\mathcal{S}^{\perp})_{\oplus}$ *with* $\pi \preceq \widetilde{\pi}$, → *Use cocircuits*

(3) *and* $(\mathcal{S}, \widetilde{\mathcal{S}})$ *is nondegenerate.* → *???*

→ Recursive algorithm for (3) [AMR24]

- Nonnegative cocircuits
- Linear inequality systems

Summary & Outlook

- Packages:
  https://github.com/MarcusAichmayr
- C++ libraries
- CertLin paper in preparation [AR26b]
- Implement more sign vector conditions for CRNs
- Study conditions for positive solutions of polynomial equations

References

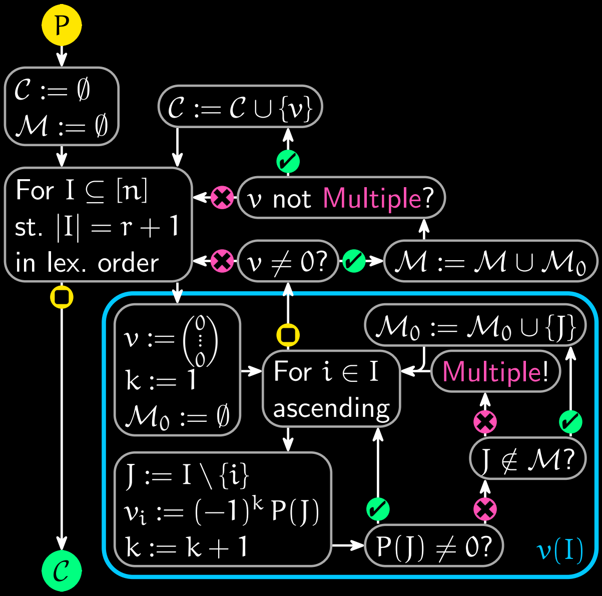
[Aic26] Marcus S. Aichmayr. "SignVectors: a SageMath Package for Oriented Matroids". Accepted. Software presentation ISSAC. 2026.

[AMR24] Marcus S. Aichmayr, Stefan Müller, and Georg Regensburger. "A SageMath package for elementary and sign vectors with applications to chemical reaction networks". In: Mathematical software—ICMS 2024. Vol. 14749. LNCS. Springer, Cham, 2024, pp. 155–164.

[AR26a] Marcus S. Aichmayr and Georg Regensburger. "Computing Circuits over Integral Domains". Submitted. 2026.

[AR26b] Marcus S. Aichmayr and Georg Regensburger. "How to Certify Solvability of Linear Inequality Systems with Circuits". In preparation. 2026.

Unique Enumeration of Circuits



Homogeneous Linear Inequality Systems

Theorem (Motzkin 1936)

Exactly one of the following has a solution:

(H1) $\boxed{\begin{array}{c} Ax > 0 \\ Bx \geq 0 \\ Cx = 0 \end{array}}$ (H2) $\boxed{\begin{array}{c} A^\top u + B^\top v + C^\top w = 0 \\ u, v \geq 0, \qquad u \neq 0 \end{array}}$

$\rightarrow$ $Mx \in \mathcal{I}$

(H1') $\boxed{\begin{pmatrix} A \\ B \\ C \end{pmatrix} x \in (0, \infty)^p \times [0, \infty)^q \times \{0\}^r}$

(H2') $\boxed{\begin{pmatrix} A^\top & B^\top & C^\top \\ \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 1^\top & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \{0\}^n \times [0, \infty)^{p+q} \times (0, \infty)}$

Homogenization of Linear Inequality Systems

Lemma (Homogenization)

The following are equivalent:

(1) $\boxed{\begin{array}{r} Ax < a \\ Bx \leq b \end{array}}$

(2) $\boxed{\begin{array}{r} \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \tau \end{pmatrix} > 0 \\ \begin{pmatrix} B & b \end{pmatrix} \begin{pmatrix} x \\ \tau \end{pmatrix} \geq 0 \end{array}}$

Proof.

If x is a solution of (1), then $(-x, 1)^\top$ is a solution of (2).
If $(x, \tau)^\top$ is a solution of (2), then $-\frac{x}{\tau}$ is a solution of (1). $\square$

Exactly One CBE – Recursive Algorithm

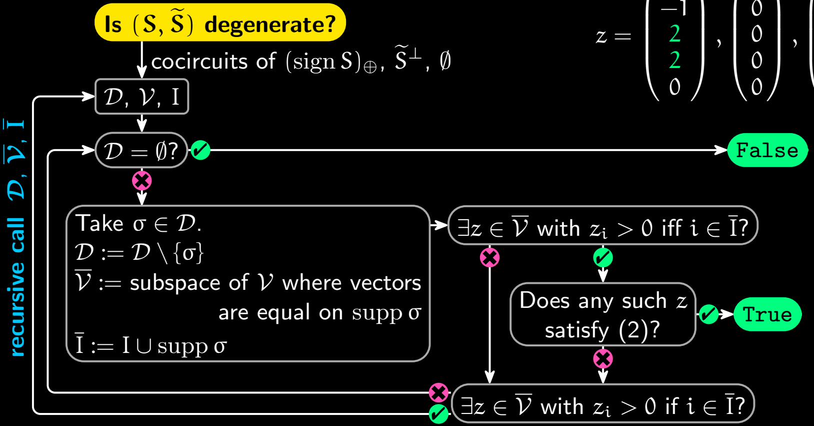
Definition

$(\mathcal{S}, \widetilde{\mathcal{S}})$ is degenerate if there is $z \in \widetilde{\mathcal{S}}^{\perp}$:

(1) There is nonempty set of cocircuits $\mathcal{D} \subseteq \operatorname{sign}(\mathcal{S})_{\oplus}$:
- For all $\sigma \in \mathcal{D}$ and for all $i, j \in \operatorname{supp} \sigma$, $z_i = z_j > 0$.
- For all $i \notin \bigcup_{\sigma \in \mathcal{D}} \operatorname{supp} \sigma$, $z_i \leq 0$.

(2) For every cocircuits $\sigma \in \operatorname{sign}(\mathcal{S}^{\perp})_{\oplus}$, $\operatorname{supp} \sigma \not\subseteq \operatorname{supp} z$.

$$z = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \quad \overbrace{\begin{pmatrix} + \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ + \\ + \\ 0 \end{pmatrix}}^{\in \mathcal{D}}$$



Package CertLin

```
sage: from certlin import *
sage: M = random_matrix(ZZ, 15, 5)
sage: I = Intervals.random(15, ring=ZZ)
sage: S = LinearInequalitySystem(M, I)
sage: S
[ 0  5  1 -1  1]  x in [1, 2)
[-1  1  0  1  0]  x in [0, 2]
[ 0  2  3 -3  0]  x in [1, 3]
[-1 -5  0  0  1]  x in (-1, 0]
[-1 -5 -1  0  3]  x in (-oo, -1)
[ 0  1 -5 -1  1]  x in [1, +oo)
[-2  1  1  1  0]  x in (-1, 0]
[ 2  1 -1 -1  1]  x in (-oo, +oo)
[-1  1  0 -4 -1]  x in (-2, -1)
[ 1  0 -2  1  1]  x in (0, 1]
[ 4 -3  3 -3  0]  x in [-1, 1)
[-1  1  0  2  1]  x in (-1, +oo)
[ 1  0 -2 -2  1]  x in (-oo, 1)
[-1 -2 -4  0 -3]  x in {0}
[ 1 -4 -2 -1  0]  x in (-oo, -2]
```

CertLin: Certify

```
sage: S.certify_unsolvability() # sparse
(134, 0, 0, 356, -216, 14, 0, 0, 0, 144, 0, 0, 0, 0, -4)

sage: S.certify() # both algorithms
(False, (134, 0, 0, 356, -216, 14, 0, 0, 0, 144, 0, 0, 0, 0, -4))

sage: S.certify(random=True) # faster
(False, (0, -1, 9, 0, 0, 25, -12, 0, -30, -55, 0, 0, 0, 0, 0))
```

CertLin: Homogenization

```
sage: S.to_homogeneous()
[ 0  5  1 -1  1 -2]  x >  0
[ 1  5  0  0 -1 -1]  x >  0
[-1 -5 -1  0  3  1]  x >  0
[ 2 -1 -1 -1  0 -1]  x >  0
[ 1 -1  0  4  1 -2]  x >  0
[-1  1  0 -4 -1  1]  x >  0
[-1  0  2 -1 -1  0]  x >  0
[ 4 -3  3 -3  0 -1]  x >  0
[ 1 -1  0 -2 -1 -1]  x >  0
[ 1  0 -2 -2  1 -1]  x >  0
[ 0  0  0  0  0 -1]  x >  0
[ 0 -5 -1  1 -1  1]  x >= 0
[ 1 -1  0 -1  0  0]  x >= 0
[-1  1  0  1  0 -2]  x >= 0
[ 0 -2 -3  3  0  1]  x >= 0
[ 0  2  3 -3  0 -3]  x >= 0
[-1 -5  0  0  1  0]  x >= 0
[ 0 -1  5  1 -1  1]  x >= 0
[-2  1  1  1  0  0]  x >= 0
[ 1  0 -2  1  1 -1]  x >= 0
[-4  3 -3  3  0 -1]  x >= 0
[ 1 -4 -2 -1  0  2]  x >= 0
[-1 -2 -4  0 -3  0]  x =  0
```

CertLin: Dual System

```
sage: S.dual()
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  1  1  1  1  1  1  1  1  1  1  1]   x >  0
[ 1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0]   x >= 0
[ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1]   x >= 0
[ 0  1 -1  0  0 -1  0 -2  1 -4  1 -1  1  0  1 -1  2  1 -1 -1  4  1  1  0]   x =  0
[-5 -1  1 -2  2 -5 -1  1  0  3  2 -2 -4  5  5 -5 -1 -1  1  0 -3 -1  0  0]   x =  0
[-1  0  0 -3  3  0  5  1 -2 -3  4 -4 -2  1  0 -1 -1  0  0  2  3  0 -2  0]   x =  0
[ 1 -1  1  3 -3  0  1  1  1  3  0  0 -1 -1  0  0 -1  4 -4 -1 -3 -2 -2  0]   x =  0
[-1  0  0  0  0  1 -1  0  1  0  3 -3  0  1 -1  3  0  1 -1 -1  0 -1  1  0]   x =  0
[ 1  0 -2  1 -3  0  1  0 -1 -1  0  0  2 -2 -1  1 -1 -2  1  0 -1 -1 -1 -1]   x =  0
```

CertLin: Real Algebraic Numbers

```
sage: M = matrix([[1, sqrt(3), -1/2, 0], ...])
sage: I = Intervals.from_bounds([1, -oo, ...], ...)
sage: S = LinearInequalitySystem(M, I)
sage: S
[       1   sqrt(3)      -1/2         0]  x in [1, sqrt(5))
[       1         0  -sqrt(5)         3]  x in (-oo, 0)
[ 3^(1/3)         0       5/2         1]  x in [sqrt(2), +oo)

sage: S.certify_unsolvability() # instant
...
ValueError: A solution exists.

sage: S.certify()
(True, (0, 1/30*(12*sqrt(5)*3^(5/6)*sqrt(2) + 58*sqrt(5)*3^(5/6) + 85*sqrt(5)*
     sqrt(3) + 30*sqrt(3)*sqrt(2) + 170*3^(5/6) + 145*sqrt(3))/(6*sqrt(5)
     *3^(1/3) + 2*sqrt(5) + 4*3^(1/3) + 15), 2/5*(6*sqrt(5)*3^(1/3)*sqrt(2) + 4*
     sqrt(5)*3^(1/3) + 15*sqrt(2) + 10)/(6*sqrt(5)*3^(1/3) + 2*sqrt(5) +
     4*3^(1/3) + 15), 1/2*(4*sqrt(5)*sqrt(2) - 2*sqrt(5)*3^(1/3) + 8*3^(1/3)*
     sqrt(2) + 2*sqrt(5) + 4*3^(1/3) - 5)/(6*sqrt(5)*3^(1/3) + 2*sqrt(5) +
     4*3^(1/3) + 15)))

sage: S.find_solution()
(0, 1/30*(12*sqrt(5)*3^(5/6)*sqrt(2) + 58*sqrt(5)*3^(5/6) + 85*sqrt(5)...
```

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